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Discrete filtering of numerical solutions to hyperbolic conservation laws

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SUMMARY

In the framework of finite volume approximations to the Euler equations of gas dynamics we introduce computationally cheap difference schemes in addition with efficient discrete filter operators correcting discrete values locally. After presentation of a classical discrete filter algorithm we describe for the first time the implementation of a TV filter, originally developed in signal and image processing, in the context of hyperbolic conservation laws on unstructured grids. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: discrete filtering; TV filter; image processing

1. INTRODUCTION

We consider scalar spatially two-dimensional conservation laws

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$$

for functions $u : \mathbb{R}^2 \times \mathbb{R}_+ \ni x, t \mapsto u(x,t) \in D \subset \mathbb{R}$ where f and g are the flux functions. Systems of this kind are known to exhibit discontinuities even if the initial condition is arbitrarily smooth. Therefore, one has to consider weak solutions, for example of type $u \in BV([0,t] \rightarrow L^\infty \cap L^1(\mathbb{R}^2))$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\sigma u \,\mathrm{d}x \,\mathrm{d}y + \oint_{\partial\sigma} f(u) n_x + g(u) n_y \,\mathrm{d}s = 0$$

for all bounded control volumes $\sigma \subset \mathbb{R}^2$ or an equivalent definition. Here, $n = (n_x, n_y)^{\mathrm{T}}$ denotes the unit outer normal vector at σ.

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The simplest case of a numerical approximation of weak solutions is the finite difference scheme on a cartesian mesh with mesh sizes Δx and Δy given by

$$\frac{u_{i,j}(t+\Delta t)-u_{i,j}(t)}{\Delta t} = -\frac{1}{\Delta x}(F(u_{i+1,j},u_{i,j})-F(u_{i,j},u_{i-1,j})) - \frac{1}{\Delta y}(G(u_{i,j+1},u_{i,j})-G(u_{i,j},u_{i,j-1}))$$

where $F_{i+1/2,j}:=F(u_{i+1,j},u_{i,j})$ and $G_{i,j+1/2}:=G(u_{i,j},u_{i,j+1})$ are numerical flux functions supposed to satisfy the consistency condition $F(s,s)=f(s)$ and $G(s,s)=g(s)$, respectively. It was shown by Tadmor that any numerical flux of this type can be written uniquely in the viscosity form

$$F_{i+1/2,j} = \frac{1}{2}(f(u_{i,j})+f(u_{i+1,j})) - \frac{\Delta x}{2\Delta t}Q(u_{i+1,j},u_{i,j})(u_{i+1,j}-u_{i,j})$$

where $Q_{i+1/2,j}:=Q(u_{i+1,j},u_{i,j})$ is the viscosity coefficient uniquely characterizing each of these difference schemes. Note that the unconditionally unstable central difference corresponds to

$$F^{\mathrm{c}}_{i+1/2,j}:=\tfrac{1}{2}(f(u_{i,j})+f(u_{i+1,j}))$$

and that the stabilizing viscosity term can be interpreted as the discretization of a second derivative $\partial_x(D(u)\partial_x u)$ for some viscosity function D depending on Q (and analogously in y-direction for G). Hence, the finite difference method is a discretization of a perturbed equation

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = \varepsilon\nabla\cdot(D(u)\nabla u)$$

where ε contains powers of $\Delta t, \Delta x, \Delta y$.

Nowadays sophisticated difference schemes like TVD [1], ENO [2, 3] or WENO [4] methods rely on non-linear algorithms defining the diffusion tensor D implicitly.

In contrast, there exists a various number of non-linear diffusion equations in the area of image processing, see Reference [5], where we can find models of equations with positive diffusion coefficients while diffusing backwards or efficient total variation preserving discrete filters. The use of these results in the framework of conservation laws can be shown by considering computationally cheap but oscillating high-order difference schemes in conjunction with such a discrete filter operator. These methods lead to simple and fast algorithms capable of eliminating spurious oscillations near shocks without smearing the shocks themselves.

In computational fluid dynamics first ideas in this area are developed by Engquist *et al.* [6] who designed a discrete filter to convert a simple second-order scheme into a TVD scheme. Therefore, in the first section of our paper, we want to describe one of their filter algorithms. By converting the discrete filter to the continuous form we reveal its close relationship to a continuous TV model developed in the area of image processing. Because the total variation may be the key to new dissipation models, we introduce a discrete TV filter due to Chan *et al.* [7] in Section 3. For the first time, we will describe its implementation in the framework of finite volume approximation to the Euler equations of gas dynamics on unstructured grids. Numerical results with remarkable properties are presented.

2. DISCRETE FILTER OPERATORS

In 1989 Engquist *et al.* have published an essay about 'non-linear filters for efficient shock computation' [6] in which they presented several discrete filter operators for the numerical approximation of hyperbolic conservation laws with discontinuous solutions.

For simplicity they considered spatially one-dimensional conservation laws

$$\partial_t u + \partial_x f(u) = 0, \quad u(x,0) = u_0(x)$$

with a given finite difference scheme G, so that $u^{n+1} = G(u^n)$. By processing the discrete solution with a non-linear conservation form filter P at every time step,

$$\tilde{u}^{n+1} = P(u^{n+1}, u^n)$$

they obtain simple and fast algorithms, capable of eliminating spurious oscillations near shocks without smearing the shock itself.

The most simple form Engquist, Lötstedt and Sjögreen suggested for P if there is an extremum in point j and if $|\Delta_- u_j| > |\Delta_+ u_j|$, is

$$\tilde{u}_j = u_j + \text{sgn}(\Delta_+ u_j) \cdot \min\left(|\Delta_+ u_j|, \frac{|\Delta_- u_j|}{2}\right) \tag{1}$$

with $\Delta_+ u_j = u_{j+1} - u_j$ and $\Delta_- u_j = -(u_{j-1} - u_j)$.

In order to transform this discrete filter into a continuous form, we introduce an artificial time scale $\Delta\tau$

$$\frac{\tilde{u}_j - u_j}{\Delta\tau} = \frac{1}{\Delta\tau} \text{sgn}(\Delta_+ u_j) \cdot \min\left(|\Delta_+ u_j|, \frac{|\Delta_- u_j|}{2}\right) \tag{2}$$

where the left-hand side already represents the continuous term $\partial_\tau u\,|_j$.

The expressions $|\Delta_+ u_j|$ and $|\Delta_- u_j|$ can be transformed by means of a Taylor expansion

$$\begin{aligned} |\Delta_+ u_j| &= |u_{j+1} - u_j| \approx |u(x_{j+1}) - u(x_j)| = |u(x+h) - u(x)| \\ &= \left|hu'(x) + \frac{h^2}{2} u''(x) + O(h^3)\right| \leqslant |u'(x)|h + \left|\frac{h^2}{2} u''(x) + O(h^3)\right| \end{aligned}$$

and analogously

$$|\Delta_- u_j| \leqslant |u'(x)|h + \left|-\frac{h^2}{2} u''(x) + O(h^3)\right|$$

That means for the minimum in (2)

$$\begin{aligned} &\min\left(|\Delta_+ u_j|, \frac{|\Delta_- u_j|}{2}\right) \\ &\leqslant \min\left(|u'(x)|h + \left|\frac{h^2}{2} u''(x) + O(h^3)\right|, |u'(x)|\frac{h}{2} + \frac{1}{2}\left|-\frac{h^2}{2} u''(x) + O(h^3)\right|\right) \end{aligned}$$

Because terms of order less or equal h^2 are negligible, it follows that:

$$\min\left(|\Delta_+ u_j|, \frac{|\Delta_- u_j|}{2}\right) = |u'(x)| \cdot \min\left(h, \frac{h}{2}\right) + O(h^2)$$
$$= |\nabla_x u| \cdot \frac{h}{2} + O(h^2) \ (h > 0)$$

and

$$\text{sgn}(\Delta_+ u_j) = \text{sgn}(\nabla_x u \cdot h + O(h^2))$$

Hence, with the notation $F(\cdot) = (h/2\Delta\tau)\,\text{sgn}(\cdot)$ and the assumption $\text{sgn}(\Delta_+ u_j) = \text{sgn}(\nabla_x u \cdot h)$ the approximated continuous form of the discrete filter (1) is

$$\partial_\tau u = |\nabla_x u| \frac{h}{2\Delta\tau} \text{sgn}(\nabla_x u \cdot h)$$
$$= |\nabla_x u| F(\nabla_x u \cdot h) \tag{3}$$

In the context of image processing, we can find a definition of TV-preserving methods of filtering [5] as the steady-state solution of the problem

$$\partial_t u = -|\nabla u| F(\mathscr{L}(u)), \quad u(x, 0) = u_0(x) \tag{4}$$

with $\text{sgn}(F(u)) = \text{sgn}(u)$, and $\mathscr{L}(u)$ is a second-order elliptic operator whose zero-crossings correspond to edges. The signal u_0 is the noisy signal we start with.

If we compare this equation with expression (3) we notice the main difference (apart from the negative sign) in the operator $\mathscr{L}(u)$. Contrary to the postulated properties in the definition, $\mathscr{L}(u)$ in (3) equals the first-order operator $\nabla u \cdot h$. This can be traced back to the fact that the filter in (3) is not TV-preserving, not to speak about TV-diminishing.

Now the question arises if we can modify the filter (3) on the basis of definition (4) to a TV-preserving one. Therefore, it is possible to replace the first-order operator $\nabla u \cdot h$ by a second-order one, for example the Laplace operator Δu multiplied with h. This means we need a discrete term whose continuous version equals $\Delta u \cdot h$. With the help of Taylor expansion we can show that the expression $\Delta_+ u_j'$ fulfils this condition and in consideration of the negative sign postulated in definition (4) the discrete filter (1) becomes

$$\tilde{u}_j = u_j - \text{sgn}(\Delta_+ u_j') \cdot \min\left(|\Delta_+ u_j|, \frac{|\Delta_- u_j|}{2}\right) \tag{5}$$

with $\Delta_+ u_j' = (u_{j+1} - 2u_j + u_{j-1})/h^2$ as possible discretization.

In order to obtain a reasonable filter, the possibility that an extremum at x_j is followed by one at x_{j+1} must be excluded, because the signum function is not defined for $\Delta_+ u_j' = 0$.

It is remarkable that Engquist *et al.* have made the same restriction (that there are no consecutive extrema) in their more sophisticated TVD filter algorithms in Reference [6].

Therefore, it seems reasonable to use results achieved in the context of image processing in order to derive discrete filter operators for numerical solutions to hyperbolic conservation laws.

3. A DIGITAL TV FILTER FROM IMAGE PROCESSING

Due to the fact that the total variation of a numerical scheme for conservation laws plays a decisive role we have turned our attention to the theory of TV filters established in the thesis of Rudin [8] and developed by Osher and his co-workers. As a result Chan *et al.* have presented a finite difference filter on graphs capable of denoising digital images without blurring jumps or edges in a recent paper [7]. In the following, we will show how this discrete model can be implemented to give cheap but proper non-linear dissipation to be used in a very simple CFD code.

Chan *et al.* consider an undirected graph with a finite set Ω of nodes and an edge dictionary E. Now, $u:\Omega\rightarrow\mathbb{R}$ is a digital signal and the value at node i is denoted by u_i. At node i the *discrete local variation*

$$|\nabla_i u| = \sqrt{\sum_{j\sim i}(u_j - u_i)^2}$$

and for any positive number ε the *regularized variation*

$$|\nabla_i u|_\varepsilon = \sqrt{|\nabla_i u|^2 + \varepsilon^2}$$

is introduced whereas $j\sim i$ means that j is a neighbour of i.

For a signal $u^0(x)$ which is assumed to be the random noise contaminated version of a clean signal, their data-dependent digital TV filter $\mathcal{F}^{\lambda,\varepsilon}: u\rightarrow v$ looks like

$$v_i = \mathcal{F}_i^{\lambda,\varepsilon}(u) = \sum_{j\sim i} h_{ij}(u)u_j + h_{ii}(u)u_i^0 \tag{6}$$

for any node i, any existing signal u and output v. The filter coefficients are given by

$$h_{ij}(u) = \frac{w_{ij}(u)}{\lambda + \sum_{k\sim i} w_{ik}(u)}, \qquad h_{ii}(u) = \frac{\lambda}{\lambda + \sum_{k\sim i} w_{ik}(u)} \tag{7}$$

with

$$w_{ij}(u) = \frac{1}{|\nabla_i u|_\varepsilon} + \frac{1}{|\nabla_j u|_\varepsilon} \tag{8}$$

and at any node i it holds $h_{ii} + \sum_{j\sim i} h_{ij} = 1$.

This non-linear low-pass filter contains two tunable parameters, a small positive regularization parameter ε for computational stability and a positive fitting parameter λ. While ε is only a technically founded constant, the fitting parameter λ is important for the restoration effect. If σ^2 denotes the variance of the assumed random noise, the size of λ is comparable to $1/\sigma^2$ and acts as an indicator for noise (which should be smoothed) or intrinsic jumps (which should not be distored by the filter).

With this background, the adaptive property of the digital filter can easily understood as follows. A large local variation $|\nabla u|$ (large enough to be distinct from noise) indicates a jump or edge inherited from u^0 and should be preserved. The filter achieves this goal since a large $|\nabla u|$ leads to a small w_{ij} (compared to λ) so that h_{ii} is nearly one and h_{ij} is nearly zero.

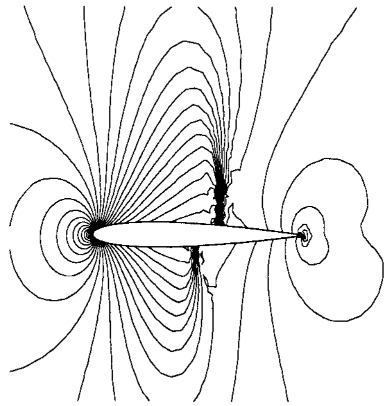

Figure 1. Lax–Wendroff solution.

This results in the preservation of the jump as expected. The case of a small local variation $|\nabla u|$ indicates a flat region with noise inherited from u^0 that should be smoothed by the filter. Indeed, a small $|\nabla u|$ effects a large w_{ij} (compared to λ) and therefore h_{ii} equals nearly zero and h_{ij} is nearly one which means smoothing of the data.

For image restoration it has been shown that in practice about 60 or 80 rounds of the TV filtering algorithm are satisfactory enough for good numerical results.

For the application of the TV filter in the context of numerical solutions of conservation laws one has to interpret the noisy signal u^0 as an oscillating solution of a hyperbolic conservation law. In contrast to the situation given in image processing we have no longer globally random noised data but local oscillations in the surrounding of jumps or shocks. To avoid errors resulting from the global variance assumption we have developed data-dependent routines for the computation of a local variance and therewith a local λ for all grid points. But our numerical results have shown, that we obtain best results with a globally setting λ much larger as required in image processing. Furthermore, instead of 60 or 80 rounds of filtering as demanded by Chan *et al.* we obtain same good results with only one single filter step acting after an oscillatory steady-state solution is reached with a simple high-order difference scheme.

For the implementation of the digital TV filter in the framework of a finite volume approximation to the Euler equations of gas dynamics on unstructured grids we choose a computationally cheap Lax–Wendroff scheme which leads to an oscillating steady-state solution. Its density is presented in Figure 1 by the use of isolines in a top view and Figure 3 illustrates a cut through this data above the airfoil. Afterwards we add one single filter step with the non-linear TV filter and in only one CPU second we get the solution shown in Figures 2 and 4. The tunable parameters are $\varepsilon = 0.0001$ and $\lambda = 140$. The sharpness of the shock is entirely preserved and the spurious oscillations are smoothed as illustrated in Figures 3 and 4. Even the oscillations around the smaller shock under the profile are smoothed by the filter without smearing the shock itself as shown by Figure 6 in comparison to the Lax–Wendroff solution in Figure 5. The digital TV filter yields smooth behaviour in continuous regions and high resolution of shocks.

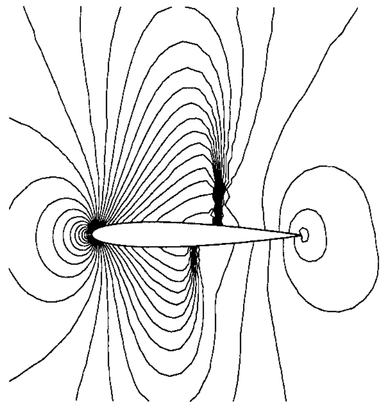

Figure 2. Solution with TV filter.

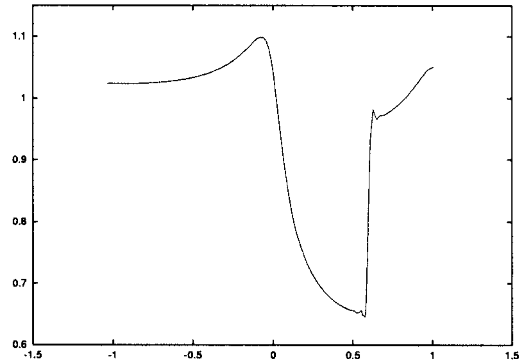

Figure 3. Lax–Wendroff solution above the airfoil.

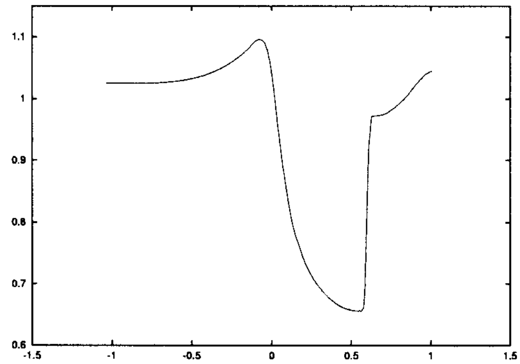

Figure 4. Filtered solution above the airfoil.

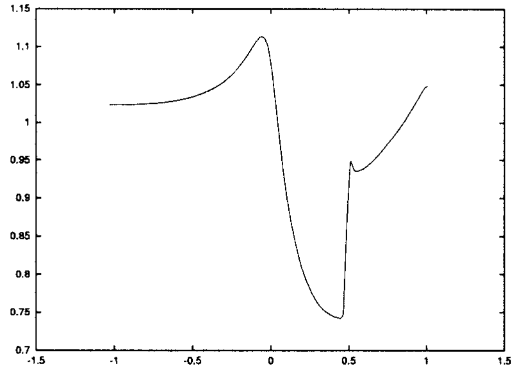

Figure 5. Lax–Wendroff solution below the airfoil.

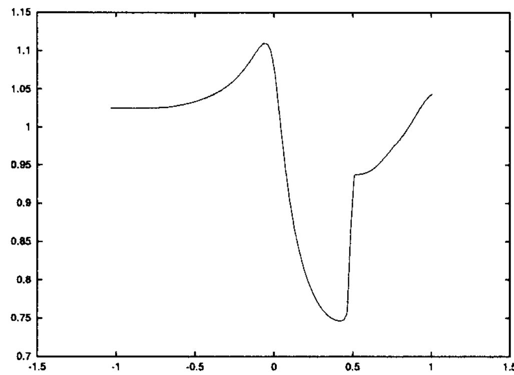

Figure 6. Filtered solution below the airfoil.

4. CONCLUSIONS

We have presented new non-linear artificial dissipation terms based on discrete filter operators originally developed in the area of image processing. While in more sophisticated difference schemes like TVD, ENO or WENO methods it is hard or even impossible to get an insight into the contained numerical dissipation the new class of methods has the promise that the dissipation is explicitly known everywhere. We hope to analyse non-linear diffusion equations further in the future in order to create whole new families of numerical schemes with controlled dissipation which also use backward diffusion in order to sharpen shocks. As another research direction we want to consider central differences of very high order which have to be equipped with fast discrete TV filters.

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